Domination related parameters in rooted product graphs

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Abstract

A set S of vertices of a graph G is a dominating set in G if every vertex outside of S is adjacent to at least one vertex belonging to S. A domination parameter of G is related to those sets of vertices of a graph satisfying some domination property together with other conditions on the vertices of G. Here, we investigate several domination related parameters in rooted product graphs.

Keywords: Domination; Roman domination; domination related parameters; rooted product graphs.

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1 Introduction

Domination in graph constitutes a very important area in graph theory [9]. An enormous quantity of researches about domination in graphs have been developed in the last years. Nevertheless, there are still several open problems and incoming researches about that. One interesting question in this area is related to the study of domination related parameters in product graphs. For instance, the Vizing's conjecture [20, 21], is one of the most popular open problems about domination in product graphs. The Vizing's conjecture states that the domination number of Cartesian product graphs is greater than or equal to the product of the domination numbers of the factor graphs. Moreover, several kind of domination related parameters have been studied in the last years. Some of the most remarkable examples are the following ones. The domination number of direct product graphs was studied in [3, 11, 16]. The total domination number of direct product graphs was studied in [5]. The upper domination number of Cartesian product graphs was studied in [2]. The independence domination number of Kronecker product graphs was studied in [10]. Several domination

related parameters of corona product graphs and the conjunction of two graphs were studied in [8] and [22], respectively. The Roman domination number of lexicographic product graphs was studied in [12]. According to the quantity of works devoted to the study of domination related parameters in product graphs it is noted that not only Vizing's conjecture is an interesting topic related to domination in product graphs. In this sense, in this paper we pretend to contribute with the study of some domination related parameters for the case of rooted product graphs.

We begin by establishing the principal terminology and notation which we will use throughout the article. Hereafter $G = (V, E)$ represents an undirected finite graph without loops and multiple edges with set of vertices V and set of edges E. The order of G is $|V| = n(G)$ and the size $|E| = m(G)$ (If there is no ambiguity we will use only n and m). We denote two adjacent vertices $u, v \in V$ by $u \sim v$ and in this case we say that uv is an edge of G or $uv \in E$. For a nonempty set $X \subseteq V$ and a vertex $v \in V$, $N_X(v)$ denotes the set of neighbors that v has in X: $N_X(v) := \{u \in X : u \sim v\}$ and the degree of v in X is denoted by $\delta_X(v) = |N_X(v)|$. In the case $X = V$ we will use only $N(v)$, which is also called the open neighborhood of a vertex $v \in V$, and $\delta(v)$ to denote the degree of v in G. The close neighborhood of a vertex $v \in V$ is $N[v] = N(v) \cup \{v\}$. The minimum and maximum degrees of G are denoted by δ and Δ, respectively. The subgraph induced by $S \subset V$ is denoted by $\langle S \rangle$ and the complement of the set S in V is denoted by $\overline{S}$. The distance between two vertices $u, v \in V$ of G is denoted by $d_G(u, v)$ (or $d(u, v)$ if there is no ambiguity).

The set of vertices $D \subset V$ is a *dominating set* of G if for every vertex $v \in \overline{D}$ it is satisfied that $N_D(v) \neq \emptyset$. The minimum cardinality of any dominating set of G is the *domination number* of G and it is denoted by $\gamma(G)$. A set D is a $\gamma(G)$-set if it is a dominating set and $|D| = \gamma(G)$. Throughout the article we follow the terminology and notation of [9].

Given a graph G of order n and a graph H with root vertex v, the rooted product $G \circ H$ is defined as the graph obtained from G and H by taking one copy of G and n copies of H and identifying the vertex u_i of G with the vertex v in the i^{th} copy of H for every $1 \leq i \leq n$ [7]. If H or G is the singleton graph, then $G \circ H$ is equal to G or H, respectively. In this sense, to obtain the rooted product $G \circ H$, hereafter we will only consider graphs G and H of orders greater than or equal to two. Figure 1 shows the case of the rooted product graph $P_4 \circ C_3$. Hereafter, we will denote by $V = \{v_1, v_2, ..., v_n\}$ the set of vertices of G and by $H_i = (V_i, E_i)$ the i^{th} copy of H in $G \circ H$.

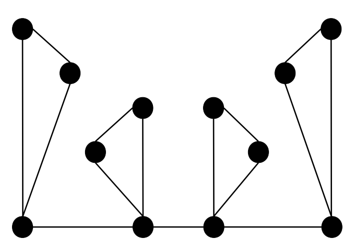

Figure 1: The rooted product graph $P_4 \circ C_3$.

It is clear that the value of every parameter of the rooted product graph depends on the root of the graph H. In the present article we present some results related to some domination parameters in rooted product graphs.

2 Domination number

We begin with the following remark which will be useful into proving next results.

Lemma 1. *Let G be a graph of order $n \geq 2$ and let H be any graph with root v and at least two vertices. If v does not belong to any $\gamma(H)$-set or v belongs to every $\gamma(H)$-set, then*

$$\gamma(G \circ H) = n\gamma(H).$$

Proof. If A_i is a dominating set of minimum cardinality in $H_i = (V_i, E_i)$, $i \in \{1, ..., n\}$, then it is clear that $\bigcup_{i=1}^{n} A_i$ is a dominating set in $G \circ H$. Thus $\gamma(G \circ H) \leq n\gamma(H)$. Suppose v does not belong to any $\gamma(H)$-set. Let S be a $\gamma(G \circ H)$-set and let $S_i = S \cap V_i$ for every $i \in \{1, ..., n\}$. Notice that the set S_i dominates all the vertices of H_i except maybe the root v_i which could be dominated by other vertex not in H_i.

If $v_j \notin S_j$ for some $j \in \{1, ..., n\}$, then S_j is a dominating set in $H_j - v_j$. So $\gamma(H_j - v_j) \leq |S_j|$. Moreover, since v_j does not belong to any $\gamma(H_j)$-set, it is satisfied that $\gamma(H_j - v_j) = \gamma(H_j)$. If $|S_j| < \gamma(H)$, then we have that $\gamma(H_j - v_j) \leq |S_j| < \gamma(H_j) = \gamma(H_j - v_j)$, a contradiction. On the other side, if $v_l \in S_l$ for some $l \in \{1, ..., n\}$, then S_l is a dominating set in H_l. So $\gamma(H_l) \leq |S_l|$. Therefore, $|S_i| \geq \gamma(H)$ for every $i \in \{1, ..., n\}$ and we obtain that $\gamma(G \circ H) = n\gamma(H)$.

On the other hand, let us suppose v belongs to every $\gamma(H)$-set. Thus, v dominates at least one vertex in H which is not dominated by any other vertex in every $\gamma(H)$-set and, as a consequence, $\gamma(H - v) \geq \gamma(H)$. As above, S denotes a $\gamma(G \circ H)$-set and $S_i = S \cap V_i$ for every $i \in \{1, ..., n\}$. If $v_j \notin S_j$ for some $j \in \{1, ..., n\}$, then either S_j is not a dominating set in H_j (which is a contradiction) or $|S_j| \geq \gamma(H_j - v_j) \geq \gamma(H_j)$. On the contrary, if $v_j \in S_j$, then S_j is a dominating set in H_j and $|S_j| \geq \gamma(H_j)$. Therefore, we have that $|S| = \sum_{i=1}^{n} |S_i| \geq \sum_{i=1}^{n} \gamma(H_i) = n\gamma(H)$ and the proof is complete. □

Theorem 2. *Let G be a graph of order $n \geq 2$. Then for any graph H with root v and at least two vertices,*

$$\gamma(G \circ H) \in \{n\gamma(H), n(\gamma(H) - 1) + \gamma(G)\}.$$

Proof. It is clear that $\gamma(G \circ H) \leq n\gamma(H)$ and also, from Lemma 1, there are rooted product graphs $G \circ H$ such that $\gamma(G \circ H) = n\gamma(H)$. Now, let us suppose that $\gamma(G \circ H) < n\gamma(H)$. Let V be the set of vertices of G and let V_i, $i \in \{1, ..., n\}$, be the set of vertices of the copy H_i of H in $G \circ H$. Hence, if S is a $\gamma(G \circ H)$-set, then there exists $j \in \{1, ..., n\}$ such that $|S \cap V_j| < \gamma(H)$. Notice that the set $S \cap V_j$ dominates all the vertices in V_j excluding v_j. If $|S \cap V_j| < \gamma(H) - 1$, then the set $(S \cap V_j) \cup \{v_j\}$ is a dominating set in H_j and $|(S \cap V_j) \cup \{v_j\}| \leq |(S \cap V_j)| + 1 < \gamma(H)$, which is a contradiction. So, $|S \cap V_i| \geq \gamma(H) - 1$ for every $i \in \{1, ..., n\}$.

Let x be the number of copies $H_{j_1}, H_{j_2}, ..., H_{j_x}$ of H in which the vertex v_{j_i} of G is not dominated by $S \cap V_{j_i}$ (*i.e.*, v_{j_i} is dominated by a vertex of G belonging to other copy H_l, with $l \notin \{j_1, ..., j_x\}$). On the contrary, let $y = n - x$ be the number of copies $H_{k_1}, H_{k_2}, ..., H_{k_y}$ of H in which the vertex v_{k_i} of G is dominated by $S \cap V_{k_i}$ or $v_{k_i} \in S$. Note that the y vertices v_{k_i} of G satisfying the above property form a dominating set in G and, as a consequence, $\gamma(G) \leq y$. Since $n = x + y$, we have that $x \leq n - \gamma(G)$. Also, notice that if the vertex v_{k_i} of G is dominated by $S \cap V_{k_i}$ or $v_{k_i} \in S$, then $S \cap V_{k_i}$ is a dominating set in H_{k_i}. So, $\gamma(H) \leq |S \cap V_{k_i}|$ for every copy H_{k_i} in which the vertex v_{k_i} of G is dominated by $S \cap V_{k_i}$ or $v_{k_i} \in S$. Thus we have the following.

$$
\begin{aligned}
\gamma(G \circ H) = |S| &= \left| \bigcup_{i=1}^{x} (S \cap V_{j_i}) \cup \bigcup_{i=1}^{y} (S \cap V_{k_i}) \right| \\
&= \sum_{i=1}^{x} |S \cap V_{j_i}| + \sum_{i=1}^{y} |S \cap V_{k_i}| \\
&\geq x(\gamma(H) - 1) + y\gamma(H) \\
&= n\gamma(H) - x \\
&\geq n\gamma(H) - n + \gamma(G) \\
&= n(\gamma(H) - 1) + \gamma(G).
\end{aligned}
$$

On the other side, let A be a $\gamma(G \circ H)$-set. Since $\gamma(G \circ H) < n\gamma(H)$, there exists at least one copy H_k of H such that $|A \cap V_k| < \gamma(H)$, which implies $|A \cap V_k| \leq \gamma(H) - 1$. Since $A \cap V_k$ dominates all the vertices of H_k except maybe the root v_k, we have that if $v_k \in A \cap V_k$, then $A \cap V_k$ is a dominating set in H, which is a contradiction. So, $v_k \notin A \cap V_k$. Now, as $|A \cap V_i| \geq \gamma(H) - 1$ for every $i \in \{1, ..., n\}$, we obtain that $|A \cap V_k| = \gamma(H) - 1$. So, $A' = (A \cap V_k) \cup \{v_k\}$ is a $\gamma(H)$-set. Let us denote by A'_i, $i \in \{1, ..., n\}$, the set of vertices of $A' - \{v_i\}$ in each copy H_i of $G \circ H$.

Let B be a $\gamma(G)$-set and let $D = (\bigcup_{i=1}^{n} A'_i) \cup B$. Since A'_i dominates the vertices of $H_i - \{v_i\}$ for every $i \in \{1, ..., n\}$ and B dominates the vertices of G, we obtain that D is a dominating set in $G \circ H$. Thus

$$
|D| = \sum_{i=1}^{n} |A'_i| + |B| = n(|A'| - 1) + |B| = n(\gamma(H) - 1) + \gamma(G).
$$

Therefore, we obtain that $\gamma(G \circ H) \leq n(\gamma(H) - 1) + \gamma(G)$ and the result follows. □

3 Roman domination number

Roman domination number was defined by Stewart in [18] and studied further by some researchers, for instance in [4]. Given a graph $G = (V, E)$, a map $f : V \to \{0, 1, 2\}$ is a *Roman dominating function* for G if for every vertex v with $f(v) = 0$, there exists a vertex $u \in N(v)$ such that $f(u) = 2$. The *weight* of a Roman dominating function is given by $f(V) = \sum_{u \in V} f(u)$. The minimum weight of a Roman dominating function on G is called the *Roman domination number* of G and it is denoted by $\gamma_R(G)$. A function f is a $\gamma_R(G)$-function in a graph $G = (V, E)$ if it is a Roman dominating function and $f(V) = \gamma_R(G)$.

Let f be a Roman dominating function on G and let B_0, B_1 and B_2 be the sets of vertices of G induced by f, where $B_i = \{v \in V \ : \ f(v) = i\}$. Frequently, a Roman dominating function f is represented by the sets B_0, B_1 and B_2, and it is common to denote $f = (B_0, B_1, B_2)$. It is clear that for any Roman dominating function f on the graph $G = (V, E)$ of order n we have that $f(V) = \sum_{u \in V} f(u) = 2|B_2| + |B_1|$ and $|B_2| + |B_1| + |B_0| = n$. The following lemmas will be useful into proving other results in this section.

Lemma 3. [4] *For any graph G, $\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$.*

Lemma 4. *Let $G = (V, E)$ be a graph and let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G)$-function. Then for every $v \in V$,*

(i) *If* $v \in B_0$*, then* $\gamma_R(G) - 1 \le \gamma_R(G - v) \le \gamma_R(G)$*.*

(ii) *If* $v \in B_1$*, then* $\gamma_R(G - v) = \gamma_R(G) - 1$*.*

(iii) *If* $v \in B_2$*, then* $\gamma_R(G) - 1 \le \gamma_R(G - v) \le \gamma_R(G) + \delta(v) - 2$*.*

Proof. Let $f' = (A_0, A_1, A_2)$ be a $\gamma_R(G - v)$-function. By making $f'(v) = 1$ we have that f' is a Roman dominating function in G. Thus

$$\gamma_R(G) \le \gamma_R(G - v) + 1. \tag{1}$$

Now, if $v \in B_0$, then it is clear that $\gamma_R(G - v) \le \gamma_R(G)$ and (i) is proved.

Moreover, if $v \in B_1$, then $(B_0, B_1 - \{v\}, B_2)$ is a Roman dominating function in $G - v$. Thus $\gamma_R(G - v) \le \gamma_R(G) - 1$. Therefore, by (1) we obtain (ii).

On the other hand, if $v \in B_2$, then $(B_0, B_1 \cup (N(v) - B_2), B_2 - \{v\})$ is a Roman dominating function in $G - v$. Thus

$$\begin{aligned} \gamma_R(G - v) &\le 2|B_2 - \{v\}| + |B_1 \cup (N(v) - B_2)| \\ &= 2|B_2| - 2 + |B_1| + |N(v) - B_2| \\ &\le \gamma_R(G) + \delta(v) - 2. \end{aligned}$$

Therefore, (iii) is proved. □

Lemma 5. *Let* $G = (V, E)$ *be a graph. If for every* $\gamma_R(G)$*-function* $f = (B_0, B_1, B_2)$ *is satisfied that* $v \in B_0$*, then*

$$\gamma_R(G - v) = \gamma_R(G).$$

Proof. From Lemma 4 (i) we have that $\gamma_R(G-v) \le \gamma_R(G)$. If $\gamma_R(G-v) < \gamma_R(G)$, then there exists a $\gamma_R(G-v)$-function $h = (A_0, A_1, A_2)$ such that $h(V - \{v\}) = \gamma_R(G-v) < \gamma_R(G)$, which leads to $h(V - \{v\}) \le \gamma_R(G) - 1$. If h' is a function in G such that for every $u \in V$, $u \ne v$, we have that $h'(u) = h(u)$ and $h'(v) = 1$, then h' is a Roman dominating function in G. Thus, $\gamma_R(G) \le h'(V) = h(V - \{v\}) + 1 \le \gamma_R(G)$. So, $\gamma_R(G) = h'(V) = h(V - \{v\}) + 1 = \gamma_R(G)$ and we have that h' is a $\gamma_R(G)$-function such that $h'(v) = 1$, which is a contradiction. Therefore, $\gamma_R(G - v) = \gamma_R(G)$. □

The Roman domination number of rooted product graphs is studied at next.

Theorem 6. *Let* G *be a graph of order* $n \ge 2$*. Then for any graph* H *with root* v *and at least two vertices,*

$$n(\gamma_R(H) - 1) + \gamma(G) \le \gamma_R(G \circ H) \le n\gamma_R(H).$$

Proof. It is clear that $\gamma_R(G \circ H) \le n\gamma_R(H)$. Let V_i be the set of vertices of H_i for every $i \in \{1, ..., n\}$ and let $f = (B_0, B_1, B_2)$ be a $\gamma_R(G \circ H)$-function. Now, for every $i \in \{1, ..., n\}$ and every $k \in \{0, 1, 2\}$, let $B_k^{(i)} = B_k \cap V_i$. Let $j \in \{1, ..., n\}$. We consider the following cases.

Case 1: $v_j \in B_0^{(j)}$. If $N_{H_j}(v_j) \cap B_2^{(j)} \ne \emptyset$, then $f_j = (B_0^{(j)} - \{v_j\}, B_1^{(j)}, B_2^{(j)})$ is a Roman dominating function in $H_j - v_j$. On the contrary, if $N_{H_j}(v_j) \cap B_2^{(j)} = \emptyset$, then v_j is adjacent to some vertex $v_k \in B_2^{(k)}$, with $k \ne j$ and, again $f_j = (B_0^{(j)} - \{v_j\}, B_1^{(j)}, B_2^{(j)})$ is a Roman dominating function in $H_j - v_j$. So, $\gamma_R(H_j - v_j) \le 2|B_2^{(j)}| + |B_1^{(j)}|$. By Lemma 4 (i) we have that $\gamma_R(H_j - v_j) \ge \gamma_R(H) - 1$. Thus $2|B_2^{(j)}| + |B_1^{(j)}| \ge \gamma_R(H) - 1$.

Case 2: $v_j \in B_1^{(j)}$. Hence, it is clear that $f_j = (B_0^{(j)}, B_1^{(j)} - \{v_j\}, B_2^{(j)})$ is a Roman dominating function in $H_j - v_j$. So, $\gamma_R(H_j - v_j) \le 2|B_2^{(j)}| + |B_1^{(j)}| - 1$. By Lemma 4 (ii) we have that $\gamma_R(H_j - v_j) = \gamma_R(H) - 1$. Thus $2|B_2^{(j)}| + |B_1^{(j)}| \ge \gamma_R(H)$.

Case 3: $v_j \in B_2^{(j)}$. Thus $f_j = (B_0^{(j)}, B_1^{(j)}, B_2^{(j)})$ is a Roman dominating function in H_j. So, $2|B_2^{(j)}| + |B_1^{(j)}| \ge \gamma_R(H)$.

Now, let V be the set of vertices of G and let $A \subseteq V \cap B_0$ be the set of vertices of G such that for every vertex $v_l \in A$ is satisfied that $N_{H_l}(v_l) \cap B_2^{(l)} = \emptyset$. So, every vertex $v_l \in A$ is dominated by some vertex in $(V - A) \cap B_2^{(k)}$, with $k \ne l$. As a consequence, $V - A$ is a dominating set and $\gamma(G) \le n - |A|$. Since $A \subseteq V \cap B_0$, it is satisfied that $|A|$ equals at most the numbers of copies H_j of H such that $2|B_2^{(j)}| + |B_1^{(j)}| \ge \gamma_R(H) - 1$ (those copies satisfying Case 1). Thus we have the following,

$$\begin{aligned}
\gamma_R(G \circ H) &= 2|B_2| + |B_1| \\
&= \sum_{i=1}^{n} (2|B_2^{(i)}| + |B_1^{(i)}|) \\
&= \sum_{i=1}^{n-|A|} (2|B_2^{(i)}| + |B_1^{(i)}|) + \sum_{i=1}^{|A|} (2|B_2^{(i)}| + |B_1^{(i)}|) \\
&\ge (n - |A|)\gamma_R(H) + |A|(\gamma_R(H) - 1) \\
&= n\gamma_R(H) - |A| \\
&\ge n(\gamma_R(H) - 1) + \gamma(G).
\end{aligned}$$

Therefore the lower bound is proved. $\square$

As the following proposition shows, the above bounds are tight.

Theorem 7. *Let G be a graph of order $n \ge 2$ and let H be a graph with root v and at least two vertices. Then,*

(i) *If for every $\gamma_R(H)$-function $f = (B_0, B_1, B_2)$ is satisfied that $f(v) = 0$, then*

$$\gamma_R(G \circ H) = n\gamma_R(H).$$

(ii) *If there exist two $\gamma_R(H)$-functions $h = (B_0, B_1, B_2)$ and $h' = (B'_0, B'_1, B'_2)$ such that $h(v) = 1$ and $h'(v) = 2$, then*

$$\gamma_R(G \circ H) = n(\gamma_R(H) - 1) + \gamma(G).$$

Proof. Let $f' = (B'_0, B'_1, B'_2)$ be a $\gamma_R(G \circ H)$-function and let V_i be the set of vertices of H_i, $i \in \{1, ..., n\}$. Now, for every $i \in \{1, ..., n\}$, let $f_i = (B_0^{(i)} = B'_0 \cap V_i, B_1^{(i)} = B'_1 \cap V_i, B_2^{(i)} = B'_2 \cap V_i)$. From Theorem 6 we have that $\gamma_R(G \circ H) \le n\gamma_R(H)$. If $\gamma_R(G \circ H) < n\gamma_R(H)$, then there exists $j \in \{1, ..., n\}$ such that $f_j(V_j) = 2|B_2^{(j)}| + |B_1^{(j)}| < \gamma_R(H)$. So $f_j = (B_0^{(j)}, B_1^{(j)}, B_2^{(j)})$ is not a Roman dominating function in H_j. If $f'(v_j) = 1$ or $f'(v_j) = 2$, then every vertex in $B_0^{(j)}$ is adjacent to a vertex in $B_2^{(j)}$ and, as a consequence, $(B_0^{(j)}, B_1^{(j)}, B_2^{(j)})$ is a Roman dominating function in H_j, which is a contradiction. So $f'(v_j) = 0$ and $f_j = (B_0^{(j)} - \{v_j\}, B_1^{(j)}, B_2^{(j)})$ is a $\gamma_R(H_j - v_j)$-function. Since $f(v) = 0$ for every $\gamma_R(H)$-function, by Lemma 5 we have that

$2|B_2^{(j)}|+|B_1^{(j)}|=\gamma_R(H-v)=\gamma_R(H)$ and this is a contradiction. Therefore, $\gamma_R(G\circ H)=n\gamma_R(H)$ and (i) is proved.

To prove (ii), for every $i\in\{1,...,n\}$ we consider two $\gamma_R(H_i)$-functions $h_i=(A_0^{(i)},A_1^{(i)},A_2^{(i)})$ and $h_i'=(B_0^{(i)},B_1^{(i)},B_2^{(i)})$ such that $h_i(v_i)=1$ and $h_i'(v)=2$, and let S be a $\gamma(G)$-set. Now, we define a function g in $G\circ H$ in the following way.

- For every vertex x belonging to a copy H_j of H such that the root $v_j\in S$ we make $g(x)=h'(x)$ (notice that $g(v_j)=2$).
- For every vertex y, except the corresponding root, belonging to a copy H_l of H such that the root $v_l\notin S$, we make $g(x)=h(x)$.
- For every root of every copy H_l satisfying the conditions of the above item we make $g(x)=0$ (note that these vertices are adjacent to a vertex w of G for which $g(w)=2$).

Since every vertex $u\in V_j$ not in G, with $g(u)=0$, is adjacent to a vertex u' such that $g(u')=2$ and also, every vertex v_l of G, with $g(v_l)=0$, is adjacent to a vertex $v_k\in S$ with $g(v_k)=2$, we obtain that g is a Roman dominating function in $G\circ H$. Thus

$$\begin{aligned}\gamma_R(G\circ H)&\le\sum_{i=1}^{|S|}(2|B_2^{(i)}|+|B_1^{(i)}|)+\sum_{i=1}^{n-|S|}(2|A_2^{(i)}|+|A_1^{(i)}|-1)\\&=|S|\gamma_R(H)+(n-|S|)(\gamma_R(H)-1)\\&=n(\gamma_R(H)-1)+|S|\\&=\gamma(G)+n(\gamma_R(H)-1).\end{aligned}$$

Therefore, (ii) follows by Theorem 6. ☐

On the other hand, we can see that there are rooted product graphs for which the bounds of Theorem 6 are not achieved.

Theorem 8. *Let G be a graph of order $n\ge 2$ and let H be a graph with root v and at least two vertices. If for every $\gamma_R(H)$-function f is satisfied that $f(v)=1$, then*

$$\gamma_R(G\circ H)=n(\gamma_R(H)-1)+\gamma_R(G).$$

Proof. Let $f=(B_0,B_1,B_2)$ be a $\gamma_R(H)$-function and let $f'=(B_0',B_1',B_2')$ be a $\gamma_R(G)$-function. Now, let us define a function h in $G\circ H$ such that if $u\ne v$, then $h(u)=f(u)$. Otherwise, $h(u)=f'(u)$. Since $f(v)=1$ for every $\gamma_R(H)$-function, it is satisfied that every vertex x of $G\circ H$ with $h(x)=0$ is adjacent to a vertex y in $G\circ H$ with $h(y)=2$. Thus h is a Roman dominating function in $G\circ H$ and we have that

$$\begin{aligned}\gamma_R(G\circ H)&\le(2|B_2'|+|B_1'|)+\sum_{i=1}^{n}(2|B_2|+|B_1|-1)\\&=n(\gamma_R(H)-1)+\gamma_R(G).\end{aligned}$$

On the other hand, let V_i, $i\in\{1,...,n\}$, be the set of vertices of the copy H_i of H in $G\circ H$ and let V be the set of vertices of G. Now, let $g=(A_0,A_1,A_2)$ be a $\gamma_R(G\circ H)$-function and for every $i\in\{1,...,n\}$ let $g_i=(A_0^{(i)}=A_0\cap V_i,A_1^{(i)}=A_1\cap V_i,A_2^{(i)}=A_2\cap V_i)$. Since the root v_i of H_i satisfies that $f(v_i)=1$ for every $\gamma_R(H_i)$-function f, we have the following cases.

Case 1: If there exists $l \in \{1, ..., n\}$ such that $g(v_l) = 2$, then g_l is a Roman dominating function in H_l, but it is not a $\gamma_R(H)$-function. Thus $\gamma_R(H_l) < 2|A_2^{(l)}| + |A_1^{(l)}|$, which leads to

$$\gamma_R(H_l) \leq 2|A_2^{(l)}| + |A_1^{(l)}| - 1 = 2|A_2^{(l)} - \{v_l\}| + |A_1^{(l)}| + 1.$$

Case 2: If there exists $j \in \{1, ..., n\}$ such that $g(v_j) = 1$, then g_j is a Roman dominating function in H_j and $g'_j = (A_0^{(j)}, A_1^{(j)} - \{v_j\}, A_2^{(j)})$ is a Roman dominating function in $H_j - v_j$. Thus, by Lemma 4 (ii), it is satisfied that

$$\gamma_R(H_j) = \gamma_R(H_j - v_j) + 1 \leq 2|A_2^{(j)}| + |A_1^{(j)} - \{v_j\}| + 1.$$

Case 3: If there exists $i \in \{1, ..., n\}$ such that $g_i(v_i) = 0$, then we have one of the following possibilities:

- g_i is not a Roman dominating function in H_i. So, v_i should be adjacent to a vertex v_j, $j \neq i$, of G such that $g_j(v_j) = 2$. Moreover, $g'_i = (A_0^{(i)} - \{v_i\}, A_1^{(i)}, A_2^{(i)})$ is a Roman dominating function in $H_i - v_i$ and by Lemma 4 (ii) it is satisfied that $\gamma_R(H_i) = \gamma_R(H_i - v_i) + 1 \leq 2|A_2^{(i)}| + |A_1^{(i)}| + 1$.
- g_i is a Roman dominating function in H_i. Since $f(v_i) = 1$ for every $\gamma_R(H_i)$-function f, we have that $g_i(V_i) > \gamma_R(H_i)$. Let f_i be a $\gamma_R(H_i)$-function. Now, by taking a function g' on $G \circ H$, such that if $v \in V_i$, then $g'(v) = f'(v)$ and if $v \notin V_i$, then $g'(v) = g(v)$, we obtain that g' is a Roman dominating function for $G \circ H$ and the weight of g' is given by

$$\begin{aligned} g'\left(\bigcup_{j=1}^{n} V_j\right) &= g\left(\bigcup_{j=1, j\neq i}^{n} V_j\right) + f_i(V_i) \\ &= g\left(\bigcup_{j=1, j\neq i}^{n} V_j\right) + \gamma_R(H_i) \\ &< g\left(\bigcup_{j=1, j\neq i}^{n} V_j\right) + g_i(V_i) \\ &= g\left(\bigcup_{j=1}^{n} V_j\right) \\ &= \gamma_R(G \circ H). \end{aligned}$$

  and this is a contradiction.

As a consequence, we obtain that if $g_i(v_i) = 0$, then g_i is not a Roman dominating function in H_i. So, every vertex v_l of G for which $g(v_l) = 0$ is adjacent to a vertex v_k, $k \neq l$, of G such that $g(v_k) = 2$ and it is satisfied that the function $g' = (X_0 = A_0 \cap V, X_1 = A_1 \cap V, X_2 = A_2 \cap V)$ is a Roman dominating function in G and $\gamma_R(G) \leq 2|X_2| + |X_1|$. Thus we have the following,

$$
\begin{aligned}
\gamma_R(G\circ H) &= 2|A_2|+|A_1| \\
&= \sum_{v_i\in X_0}(2|A_2^{(i)}|+|A_1^{(i)}|)+\sum_{v_i\in X_1}(2|A_2^{(i)}|+|A_1^{(i)}|)+\sum_{v_i\in X_2}(2|A_2^{(i)}|+|A_1^{(i)}|) \\
&= \sum_{v_i\in X_0}(2|A_2^{(i)}|+|A_1^{(i)}|)+\sum_{v_i\in X_1}(2|A_2^{(i)}|+|A_1^{(i)}-\{v_i\}|)+ \\
&\quad +\sum_{v_i\in X_2}(2|A_2^{(i)}-\{v_i\}|+|A_1^{(i)}|)+|X_1|+2|X_2| \\
&\geq \sum_{v_i\in X_0}(\gamma_R(H_i)-1)+\sum_{v_i\in X_1}(\gamma_R(H_i)-1)+\sum_{v_i\in X_2}(\gamma_R(H_i)-1)+2|X_2|+|X_1| \\
&\geq \sum_{i=1}^{n}(\gamma_R(H_i)-1)+\gamma_R(G) \\
&= n(\gamma_R(H)-1)+\gamma_R(G).
\end{aligned}
$$

Therefore the result follows. □

4 Independent domination number

A set of vertices S of a graph G is *independent* if the subgraph induced by S has no edges. The maximum cardinality of an independent set in G is called the *independence number* of G and it is denoted by $\alpha(G)$. A set S is a $\alpha(G)$-set if it is independent and $|S|=\alpha(G)$. A set of vertices D of a graph G is an *independent dominating set* in G if D is a dominating set and the subgraph $\langle D\rangle$ induced by D is independent in G [1]. The minimum cardinality of any independent dominating set in G is called the *independent domination number* of G and it is denoted by $i(G)$. A set D is a $i(G)$-set if it is an independent dominating set and $|D|=i(G)$. At next we study the independent domination number of rooted product graphs and we begin by studying the independence number.

Lemma 9. *Let v be any vertex of a graph G. If v belongs to every $\alpha(G)$-set, then $\alpha(G)\geq \alpha(G-v)+1$.*

Proof. Let S be a $\alpha(G-v)$-set. Since S is still independent in G, we have $\alpha(G)\geq |S|$. If $\alpha(G)=|S|$, then S is a $\alpha(G)$-set and $v\notin S$, a contradiction. So, $\alpha(G)\geq\alpha(G-v)+1$. □

Theorem 10. *For any graph G of order $n\geq 2$ and any graph H with root v and at least two vertices,*

(i) *If there is a $\alpha(H)$-set not containing the root v, then $\alpha(G\circ H)=n\alpha(H)$.*

(ii) *If the root v belongs to every $\alpha(H)$-set, then $\alpha(G\circ H)=n(\alpha(H)-1)+\alpha(G)$.*

Proof. Let S_i, $i\in\{1,...,n\}$, be a $\alpha(H_i)$-set not containing the root v_i. Hence, $\bigcup_{i=1}^{n} S_i$ is independent in $G\circ H$. Thus $\alpha(G\circ H)\geq n\alpha(H)$. If $\alpha(G\circ H)>n\alpha(H)$, then there exists $j\in\{1,...,n\}$ such that $|S_j|>\alpha(H)$ and S_j is independent, a contradiction. Therefore, $\alpha(G\circ H)=n\alpha(H)$.

On the other hand, suppose the root v belongs to every $\alpha(H)$-set. Let A_i be a $\alpha(H_i)$-set and let B be a $\alpha(G)$-set. Since $v_i \in A_i$ for every $i \in \{1, ..., n\}$, by taking $A = B \cup (\bigcup_{i=1}^{n} A_i - \{v_i\})$ we have that A is independent in $G \circ H$. Thus

$$\alpha(G \circ H) \geq |A| = |B| + \sum_{i=1}^{n} |A_i - \{v_i\}| = n(\alpha(H) - 1) + \alpha(G).$$

Now, let V_i, $i \in \{1, ..., n\}$, be the set of vertices of the copy H_i of H in $G \circ H$ and let V be the set of vertices of G. Let X be a $\alpha(G \circ H)$-set and let $X_i = X \cap (V_i - \{v_i\})$ for every $i \in \{1, ..., n\}$ and let $Y = V \cap X$. Notice that Y and X_i are independent sets. So, $\alpha(H_i - v_i) \geq |X_i|$ and $\alpha(G) \geq |Y|$ and by Lemma 9 we have that $|X_i| \leq \alpha(H_i) - 1$. Thus

$$\alpha(G \circ H) = |Y| + \sum_{i=1}^{n} |X_i| \leq \alpha(G) + \sum_{i=1}^{n} (\alpha(H_i) - 1) = \alpha(G) + n(\alpha(H) - 1).$$

Therefore, the proof is complete. $\square$

Lemma 11. *Let $G = (V, E)$ be a graph. Then for every set of vertices $A \subset V$,*

$$i(G - A) \geq i(G) - |A|.$$

Proof. Let us suppose $i(G - A) < i(G) - |A|$. So, there exists an independent dominating set $S \subset V - A$ in $G - A$ such that $|S| < i(G) - |A|$. Let $v \in A$. If $N_S(v) \neq \emptyset$, then v is independently dominated by the set S in G. On the contrary, if $N_S(v) = \emptyset$, then the set $S \cup \{v\}$ is still independent. So, by adding those vertices which maintain the independence in the set S we obtain a set S' which is independent and dominating in G and we have that $i(G) \leq |S'| \leq |S| + |A| < i(G) - |A| + |A| = i(G)$, which is a contradiction. Therefore, $i(G - A) \geq i(G) - |A|$. $\square$

Lemma 12. *If v does not belong to any $i(G)$-set, then*

$$i(G - v) = i(G).$$

Proof. Let S be an $i(G)$-set. Since $v \notin S$, S is still independent and dominating in $G - v$. So, $i(G - v) \leq i(G)$. On the other hand, let A be an $i(G - v)$-set. Let us suppose that $|A| < i(G)$. So, $|A| \leq i(G) - 1$. If $N_A(v) = \emptyset$ in G, then $A \cup \{v\}$ is independent and dominating in G. So, $i(G) \leq |A \cup \{v\}| = |A| + 1 \leq i(G)$. Thus $|A + \{v\}| = i(G)$ and this is a contradiction because v does not belong to any $i(G)$-set. On the contrary, if $N_A(v) \neq \emptyset$, then A is independent and dominating in G, which is a contradiction ($|A| < i(G)$). So, $|A| \geq i(G)$. Therefore, $i(G - v) = |A| \geq i(G)$ and the result follows. $\square$

Theorem 13. *Let $G = (V, E)$ be a graph of order $n \geq 2$ and let H be a graph with root v and at least two vertices. Then*

$$n(i(H) - 1) + i(G) \leq i(G \circ H) \leq i(H)\alpha(G) + i(H - v)(n - \alpha(G)).$$

Proof. Let S be an $i(G \circ H)$-set and let $S_i = S \cap V_i$, $i \in \{1, ..., n\}$. If $v \in S_j$ for some $j \in \{1, ..., n\}$, then S_j is an independent dominating set in H_j. So, $|S_j| \geq i(H)$. On the contrary, if $v \notin S_k$ for some $k \in \{1, ..., n\}$, then S_k independently dominates all vertices of $H_k - v$. So, S_k is an independent dominating set in $H_k - v$ and by Lemma 11 we have

that $|S_k| \geq i(H_k - v) \geq i(H) - 1$. If $|S_j| = i(H_j) - 1$ for some $j \in \{1, ..., n\}$, then v is not independently dominated by S_j. Also, if v is independently dominated by S_l for some $l \in \{1, ..., n\}$, then $|S_l| \geq i(H_l)$. Let $A = S \cap V$ and let $B \subset V$ be the set of vertices of G such that every vertex $u_i \in B$ is independently dominated by a vertex not in G. Notice that A is an independent dominating set in $G - B$. So, by Lemma 11 we have that $|A| \geq i(G - B) \geq i(G) - |B|$ and so, $|B| \geq i(G) - |A|$. Also, for every vertex $u_i \in B$ we have that $|S_i| \geq i(H_i)$ and we have the following,

$$\begin{aligned}
|S| &= \sum_{i=1}^{n} |S_i| \\
&= \sum_{i=1}^{|A|} |S_i| + \sum_{i=1}^{|B|} |S_i| + \sum_{i=1}^{n-|A|-|B|} |S_i| \\
&\geq \sum_{i=1}^{|A|} i(H) + \sum_{i=1}^{|B|} i(H) + \sum_{i=1}^{n-|A|-|B|} (i(H) - 1) \\
&= |A|i(H) + |B|i(H) + (n - |A| - |B|)(i(H) - 1) \\
&= n(i(H) - 1) + |A| + |B| \\
&\geq n(i(H) - 1) + i(G).
\end{aligned}$$

Therefore, the lower bound follows.

To obtain the upper bound, let A be an independent set of maximum cardinality in G. Now, for every vertex $u_i \in A$ let A_i be an independent dominating set in H_i. Also, for every $u_j \notin A$ let B_j be an independent dominating set in $H_j - v$. Then, it is clear that $\left(\bigcup_{i=1}^{|A|} A_i\right) \cup \left(\bigcup_{j=1}^{n-|A|} B_j\right) \cup A$ is an independent dominating set in $G \circ H$. Therefore the upper bound follows. □

Notice that the above bounds are tight. For instance, if G is the path graph P_n and H is the star graph $S_{1,m}$, $m \geq 2$, with root v in the central vertex, (notice that $G \circ H$ is a caterpillar), then by the above theorem,

$$\begin{aligned}
i(G \circ H) &\leq i(S_{1,m})\alpha(P_n) + i(\overline{K_m})(n - \alpha(P_n)) \\
&= \left\lceil \frac{n}{2} \right\rceil + m\left(n - \left\lceil \frac{n}{2} \right\rceil\right) \\
&= mn - \left\lceil \frac{n}{2} \right\rceil (m - 1).
\end{aligned}$$

On the contrary, let S be an independent dominating set in $G \circ H$, let A be the set of vertices of P_n belonging to S and let B_i, $i \in \{1, ..., n\}$, be the set of vertices of $H_i - v$ belonging to S. If there is a copy H_j of H in $G \circ H$ such that the root v of H_j belongs to S, then neither any vertex of $H_j - v$ nor any neighbor of v in G belongs to S. Moreover, if for some copy H_l of H in $G \circ H$ is satisfied that the root v of H_l does not belong to S, then

every vertex of $H_l - v$ belongs to S. Thus,

$$\begin{aligned}|S| &= |A| + \sum_{i=1}^{n-|A|} |B_{j_i}| \\ &= |A| + m(n - |A|) \\ &= mn - |A|(m-1) \\ &\geq mn - \alpha(G)(m-1) \\ &= mn - \left\lceil \frac{n}{2} \right\rceil (m-1).\end{aligned}$$

So, $i(G \circ H) = mn - \left\lceil \frac{n}{2} \right\rceil (m-1)$ and the upper bound is tight. To see the sharpness of the lower bound, consider G as a path graph P_n and the graph H obtained from the star graph $S_{1,m}$, $m \geq 2$, by subdividing an edge. Let v be the vertex of H having distance two from the central vertex of the star. If v is the root of H, then Theorem 13 (ii) leads to

$$i(G \circ H) \geq n(i(H) - 1) + i(G) = n(2-1) + \left\lceil \frac{n}{3} \right\rceil = \left\lceil \frac{n}{3} \right\rceil + n.$$

On the other side, let A be the set of all central vertices of all copies of the star $S_{1,m}$, used to obtain $G \circ H$. Since $i(H) = 2$ we have that $|A| = n(i(H) - 1)$. Let B be an independent dominating set in the path P_n. It is clear that $A \cup B$ is an independent dominating set in $G \circ H$. So, $i(G \circ H) \leq n(i(H) - 1) + i(G) = n + \left\lceil \frac{n}{3} \right\rceil$. As a consequence $i(G \circ H) = n + \left\lceil \frac{n}{3} \right\rceil$ and the lower bound of Theorem 13 is achieved.

Moreover, notice that there are graphs in which are not attained any one of the above bounds. The next theorem is an example of that. In order to present such a result we need to introduce some notation. Let D be a set of vertices of a graph G, and let $v \in D$. We say that a vertex x is a *private neighbor* of v with respect to D if $N[x] \cap D = \{v\}$. The *private neighbor set of* v with respect to D is $pn[v, D] = N[v] - N[D - \{v\}]$.

Theorem 14. *Let $G = (V, E)$ be a graph of order $n \geq 2$ and let H be a graph with root v and at least two vertices. Then,*

(i) *If v does not belong to any $i(H)$-set, then $i(G \circ H) = ni(H)$.*

(ii) *If v belongs to every $i(H)$-set S, then*

$$i(G \circ H) \leq \alpha(G)i(H) + (n - \alpha(G))(|pn[v, S]| + i(H) - 1).$$

Proof. (i) Let us suppose v does not belong to any $i(H)$-set. From Lemma 12 we have that $i(H - v) = i(H)$. So, Theorem 13 leads to $i(G \circ H) \leq ni(H)$. Let S' be a $i(G \circ H)$-set such that $|S'| < ni(H)$. So, there exists at least one copy H_j of H such that $S_j = V_j \cap S'$ and $|S_j| < i(H)$. Since S_i independently dominates $V_i - v$ for every $i \in \{1, ..., n\}$, we have that v is not dominated by S_j in H_j. Thus, S_j is an independent dominating set in $H_j - v$ and $i(H_j - v) \leq |S_j| < i(H)$, which is a contradiction, since $i(H - v) = i(H)$. Therefore, $i(G \circ H) \geq ni(H)$ and the result follows.

(ii) Let B_i be an $i(H_i)$-set, $i \in \{1, ..., n\}$ and let C be an independent set of maximum cardinality in G. Let $S = \bigcup_{i=1}^{\alpha(G)} B_i \cup \bigcup_{j=1}^{n-\alpha(G)} (pn[v, B_j] + B_j - \{v\})$. We will show that S is an independent dominating set of $G \circ H$.

Let $B = \bigcup_{i=1}^{n} B_i$. Notice that B is a dominating set in $G \circ H$. If $G = \overline{K_n}$, then B is also independent set in $G \circ H$. In this case $\alpha(G) = n$ and the upper bound follows. Now let us suppose that $G \not\cong \overline{K_n}$. Since the root of every copy of H belongs to B, there exists at least two roots v_i and v_j, $i \neq j$, which are adjacent in $G \circ H$. Thus B is not independent in $G \circ H$.

So, $B' = \bigcup_{i=1}^{\alpha(G)} B_i \cup \bigcup_{\alpha(G)+1}^{n} (B_i - \{v\})$ is independent set in H and dominates every vertex in H_i, except $pn[v, B_i]$. Notice that $B_i - \{v\}$ is still independent in H_i, and also, it dominates every vertex in H_i, except $pn[v, B_i]$.

Therefore, we have that $i(G \circ H) \leq \alpha(G)i(H) + (n - \alpha(G))(|pn[v, S]| + i(H) - 1)$ and the upper bound follows. $\square$

5 Connected domination number and convex domination number

A set of vertices D of a graph G is a *connected* [17] (or *convex* [15]) *dominating set* in G if D is a dominating set and the subgraph induced by D, (or the set D) is connected (or convex) in G. The minimum cardinality of any connected (or convex) dominating set in G is called the *connected* (or *convex*) *domination number* of G and it is denoted by $\gamma_c(G)$ (or $\gamma_{con}(G)$). A set D is a $\gamma_c(G)$-set (or a $\gamma_{con}(G)$-set) if it is a connected (or a convex) dominating set and $|D| = \gamma_c(G)$ (or $|D| = \gamma_{con}(G)$). At next we study the connected (or convex) domination number of rooted product graphs. We begin with connected domination. This parameter was defined by Sampathkumar and Wallikar in [17].

Theorem 15. *Let G be a graph of order $n \geq 2$. Then for any graph H with root v and at least two vertices,*
$$\gamma_c(G \circ H) \in \{n\gamma_c(H), n(\gamma_c(H) + 1)\}.$$

Proof. Since the vertex v of H is a cut vertex of $G \circ H$, the vertex v of each copy H_i of H belongs to every connected dominating set of $G \circ H$. Also, the intersection of every connected dominating set of $G \circ H$ and the set of vertices of every copy of H contains a connected dominating set of H. So, $\gamma(G \circ H) \geq \sum_{i=1}^{n} \gamma_c(H) = n\gamma_c(H)$.

Hence, if v belongs to a $\gamma_c(H_i)$-set S_i, then by taking $S = \cup_{i=1}^{n} S_i$ we have that S is a connected dominating set. So, $\gamma_c(G \circ H) \leq \sum_{i=1}^{n} |S_i| = n\gamma_c(H)$. Therefore, $\gamma_c(G \circ H) = n\gamma_c(H)$.

Now, let us suppose that $\gamma_c(G \circ H) \neq n\gamma_c(H)$. So, v does not belong to any $\gamma_c(H_i)$-set S_i. Let S be a $\gamma_c(G \circ H)$-set. If $|S| < n\gamma_c(H)$, then there exists a copy H_l of H in $G \circ H$ in which $|S \cap V_l| < \gamma_c(H)$ and $S \cap V_l$ is a connected dominating set in H, which is a contradiction. So, $|S| > n\gamma_c(H)$ and there exists a copy H_j of H such that $|S \cap V_j| > \gamma_c(H)$. Since the root v of H does not belong to any $\gamma_c(H)$-set, and also v belongs to every $\gamma_c(G \circ H)$-set, we obtain that
$$|S| = \sum_{i=1}^{n} |S \cap V_i| + |V| \geq n\gamma_c(H) + n = n(\gamma_c(H) + 1).$$

On the other hand, let S_i be a $\gamma_c(H_i)$-set, $i \in \{1, ..., n\}$. Since v does not belong to any $\gamma_c(G \circ H)$-set, it is satisfied that $v \notin S_i$ for every $i \in \{1, ..., n\}$. Thus, by taking the set $S = V \cup (\bigcup_{i=1}^{n} S_i)$ we have that S is a connected dominating set and, as a consequence,
$$\gamma_c(G \circ H) \leq |S| = \sum_{i=1}^{n} |S_i| + |V| = n\gamma_c(H) + n = n(\gamma_c(H) + 1).$$

Therefore, the result follows. □

Next we study the connected domination number of some particular cases of rooted product graphs. We denote by $n_1(G)$ the number of end vertices (vertices of degree one) in G and by $\Omega(G)$ the set of end vertices in G; $|\Omega(G)| = n_1(G)$.

Lemma 16. [17] *If T is a tree of order at least three, then $\gamma_c(T) = n(T) - n_1(T)$.*

Lemma 17. *If G is a connected graph, H is a tree of order at least three and the root v is non-end vertex of H, then $\gamma_c(G \circ H) = n\gamma_c(H)$.*

Proof. Since the root of the graph H is a cut vertex in the graph $G \circ H$, we have that root of each copy H_i of H belongs to every connected dominating set of $G \circ H$. Also, the intersection of every connected dominating set of $G \circ H$ and the set of vertices of every copy of H contains a connected dominating set of H. So, $\gamma_c(G \circ H) \geq \sum_{i=1}^{n} \gamma_c(H) = n\gamma_c(H)$. Let D be a connected dominating set of $G \circ H$. Since H is a tree, from Lemma 16, no end vertex belongs to any minimum connected dominating set of H and $\gamma_c(H) = n(H) - n_1(H)$. Also, for every H_i, $\gamma_c(H_i) = n(H_i) - n_1(H_i)$. Since v is non-end vertex of H, we have that $|D| = |V(G) \cup \sum_{i=1}^{n}(V_i - \Omega_i)|$. Thus $\gamma_c(G \circ H) \leq |D| = n\gamma(H)$ and we are done. □

Lemma 18. *If T_1, T_2 are trees of order at least three, then $T_1 \circ T_2$ is also a tree of order $n(T_1 \circ T_2) = n(T_1)n(T_2)$. Moreover, $n_1(T_1 \circ T_2) \in \{n(T_1)n_1(T_2), n(T_1)(n_1(T_2) - 1)\}$.*

Proof. For a graph $T_1 \circ T_2$ is $n(T_1 \circ T_2) = n(T_1) + n(T_1)(n(T_2) - 1) = n(T_1)n(T_2)$. If a root vertex v is an end vertex of T_2, then $n_1(T_1 \circ T_2) = n(T_1)(n_1(T_2) - 1)$. On the contrary, if v is not an end vertex of T_2, then $n_1(T_1 \circ T_2) = n(T_1)n_1(T_2)$. □

Theorem 19. *Let T_1, T_2 be trees of order at least three. Then $\gamma_c(T_1 \circ T_2) = n(T_1)\gamma_c(T_2)$ if and only if the rooted vertex v of T_2 is not an end vertex of T_2.*

Proof. From Remarks 16 and 18, we have $\gamma_c(T_1 \circ T_2) = n(T_1 \circ T_2) - n_1(T_1 \circ T_2)$. Also from Lemma 18 we have $n(T_1 \circ T_2) = n(T_1)n(T_2)$. Let v be a non end vertex of T_2. Hence, $n_1(T_1 \circ T_2) = n(T_1)n_1(T_2)$. Thus $\gamma_c(T_1 \circ T_2) = n(T_1)n(T_2) - n_1(T_2)n(T_1) = n(T_1)(n(T_2) - n_1(T_2)) = n(T_1)\gamma_c(T_2)$.

Assume now $\gamma_c(T_1 \circ T_2) = n(T_1)\gamma_c(T_2)$ and suppose v is an end vertex of T_2. Hence we have $n_1(T_1 \circ T_2) = (n_1(T_2) - 1)n(T_1) = n_1(T_2)n(T_1) - n(T_1)$. Since $n(T_1 \circ T_2) = n(T_1)n(T_2)$, we have $\gamma_c(T_1 \circ T_2) = n(T_1)n(T_2) - (n_1(T_2)n(T_1) - n(T_1)) = n(T_1)(n(T_2) - n_1(T_2) + 1) = n(T_1)(\gamma_c(T_2) + 1)$, what gives a contradiction. □

From Theorems 15 and 19 we can conclude the following.

Corollary 20. $\gamma_c(T_1 \circ T_2) = n(T_1)(\gamma_c(T_2) + 1)$ *if and only if the rooted vertex v of T_2 is an end vertex of T_2.*

Convex domination was defined by Topp in [19] and it was first characterized in [15]. Notice that for the case of the convex domination number of $G \circ H$ the result is similar to Theorem 15 about connected domination. The proofs of the following results are omitted due to the analogy with the above ones.

Theorem 21. *Let G be a graph of order $n \geq 2$. Then for any graph H with root v and at least two vertices,*

$$\gamma_{con}(G \circ H) \in \{n\gamma_{con}(H), n(\gamma_{con}(H) + 1)\}.$$

Theorem 22. *If T_1, T_2 are trees, then $\gamma_{con}(T_1 \circ T_2) = n(T_1)\gamma_{con}(T_2)$ if and only if the rooted vertex v is not an end vertex of a tree T_2.*

Corollary 23. *$\gamma_{con}(T_1 \circ T_2) = n(T_1)(\gamma_{con}(T_2) + 1)$ if and only if the rooted vertex v is an end vertex of T_2.*

5.1 Weakly connected domination number

Now we consider the weakly connected domination number of rooted product graphs. A dominating set $D \subset V(G)$ is a *weakly connected dominating set* in G if the subgraph $G[D]_w = (N_G[D], E_w)$ (also called subgraph weakly induced by D) is connected, where E_w is the set of all edges having at least one vertex in D. Dunbar et al. [6] defined the *weakly connected domination number* $\gamma_w(G)$ of a graph G to be the minimum cardinality among all weakly connected dominating sets in G.

Theorem 24. *Let G be a graph of order $n \geq 2$. Then for any graph H with root v and at least two vertices,*

$$\gamma_w(G \circ H) \in \{n\gamma_w(H), n\gamma_w(H) + \gamma_w(G)\}.$$

Proof. Let D_H be a minimum weakly connected dominating set of H and D_{H_i} be the copy of D_H in the i^{th} copy H_i of $H, 1 \leq i \leq n$. Let D be a minimum weakly connected dominating set of $G \circ H$. We consider two cases.

1. $v \in D_H$. Then identified vertices belong to a minimum weakly connected dominating set of $G \circ H$ and $\gamma_w(G \circ H) = n\gamma_w(H)$.

2. $v \notin D_H$. Then $\bigcup_{i=1}^n D_{H_i} \subset D$ and identified vertices are dominated by $\bigcup_{i=1}^n D_{H_i}$. But the set $\bigcup_{i=1}^n D_{H_i}$ is not weakly connected. To make this set weakly connected, we need to add to this set $\gamma_w(G)$ vertices. So $\gamma_w(G \circ H) = |D| = |\bigcup_{i=1}^n D_{H_i}| + \gamma_w(G) = n\gamma_w(H) + \gamma_w(G)$.

□

The following lemma presented in [13] will be useful into obtaining some interesting results.

Lemma 25. *For any tree T of order $n \geq 3$,*

$$\frac{1}{2}(n - n_1(T) + 1) \leq \gamma_w(T) \leq n - n_1(T).$$

Theorem 26. *If T_1, T_2 are trees and v is not an end vertex of T_2, then*

$$\frac{1}{2}(n_1(T_1)\gamma_w(T_2) + 1) \leq \gamma_w(T_1 \circ T_2) \leq n_1(T_1)(2\gamma_w(T_2) - 1).$$

Proof. From Lemma 25, $\frac{1}{2}(n(T_1 \circ T_2) - n_1(T_1 \circ T_2) + 1) \leq \gamma_w(T_1 \circ T_2) \leq n(T_1 \circ T_2) - n_1(T_1 \circ T_2)$. Thus, from Lemma 18, we have $\frac{1}{2}(n(T_1)n(T_2) - n(T_1)n_1(T_2) + 1) \leq \gamma_w(T_1 \circ T_2) \leq n_1(T_1)(n(T_2) - n_1(T_2))$. We have $\gamma_w(T_1 \circ T_2) \leq n_1(T_1)(n(T_2) - n_1(T_2)) = n_1(T_1)2\frac{1}{2}(n(T_2) - n_1(T_2)) = n_1(T_1)2\frac{1}{2}(n(T_2) - n_1(T_2) + 1 - 1) \leq n_1(T_1)2\gamma_w(T_2) - n_1(T_1) = n_1(T_1)(2\gamma_w(T_2) - 1)$. From the other side we have $\gamma_w(T_1 \circ T_2) \geq \frac{1}{2}(n_1(T_1)(n(T_2) - n_1(T_2)) + 1) \geq \frac{1}{2}(n_1(T_1)\gamma_w(T_2) + 1)$ and finally we obtain the desired result. □

By using similar methods, we obtain the following result.

Theorem 27. *Let T_1 be a tree of order $n(T_1)$. If v is a non-end vertex of a tree T_2, then*

$$\frac{1}{2}(\gamma_w(T_2)n(T_1)+1) \leq \gamma_w(T_1 \circ T_2) \leq 2n(T_1)\gamma_w(T_2).$$

5.2 Super domination number

We continue with the super domination number of the rooted product graph. This parameter was defined in [14]. A subset D of V is called a *super dominating set* if for every $v \in V - D$ there exists $u \in N_G(v) \cap D$ such that $N_G(u) \subseteq D \cup \{v\}$. The minimum cardinality of a super dominating set is called the *super domination number* of G and it is denoted by $\gamma_{sp}(G)$. In [14] a paper was proved the following result.

Lemma 28. [14] *For any tree of order $n \geq 3$, $\frac{n}{2} \leq \gamma_{sp}(T_1 \circ T_2) \leq n - s(T)$, where $s(T)$ is the number of support vertices in T.*

Theorem 29. *Let G be a graph of order $n \geq 2$. Then for any graph H with root v and at least two vertices,*

$$\gamma_{sp}(G \circ H) = n\gamma_{sp}(H).$$

Proof. Let D_H be a minimum super dominating set of H and D_{H_i} be the copy of D_H in the i^{th} copy H_i of $H, 1 \leq i \leq n$. Let D be a minimum super dominating set of $G \circ H$. We consider two cases.

1. $v \in D_H$. Then identified vertices belong to a minimum super dominating set of $G \circ H$ and $\gamma_{sp}(G \circ H) = n\gamma_{sp}(H)$.

2. $v \notin D_H$. Then $\bigcup_{i=1}^n D_{H_i} \subset D$ and identified vertices are dominated by $\bigcup_{i=1}^n D_{H_i}$. Also, every vertex belonging to $\bigcup_{i=1}^n V_i - \bigcup_{i=1}^n D_{H_i} - U$, where U is the set of identified vertices is super dominated by $\bigcup_{i=1}^n D_{H_i}$. Suppose there exists a vertex $u \in U$ which is not super dominated by $\bigcup_{i=1}^n D_{H_i}$. Then for a vertex u does not exist any $v \in \bigcup_{i=1}^n D_{H_i}$ such that $N_{G \circ H}(v) \subseteq \bigcup_{i=1}^n D_{H_i}$. Then there exists $1 \leq i \leq n$ such that in H_i there exists a vertex which is not super dominated, a contradiction. Thus $\bigcup_{i=1}^n D_{H_i}$ is a minimum super dominating set of $G \circ H$ and $\gamma_{sp}(G \circ H) = |\bigcup_{i=1}^n D_{H_i}| = n\gamma_{sp}(H)$.

□

Now, by using Lemma 28 result, we can prove the following.

Theorem 30. *If T_1, T_2 are trees of order $n(T_1) \geq 3$ and $n(T_2) \geq 3$, respectively, then*

$$n(T_1)s(T_2) \leq \gamma_{sp}(T_1 \circ T_2) \leq n(T_1)(n(T_2) - s(T_2)).$$

Proof. If a root v is a support vertex of T_2, then $s(T_1 \circ T_2) = n(T_1) + (s(T_2) - 1)n(T_1) = n(T_1)s(T_2)$. If v is not a support vertex, then also $s(T_1 \circ T_2) = n(T_1)s(T_2)$. The upper bound holds directly from Lemma 28. For the lower bound we have $\gamma_{sp}(T_1 \circ T_2) \leq 2\gamma_{sp}(T_1 \circ T_2) - n(T_1)s(T_2)$. From this inequality we obtain the final lower bound. □

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